Prov på delkurs Diskret Matematik

26 oktober kl 8-13 MA:10S

Tillåtna hjälpmedel: Litteratur, egna anteckningar

Tentamen består av 3 uppgifter, varje uppgift kan ge 10 poäng. Deluppgifternas poäng anges inom [hakparenteser]. Observera att poänggivningen inte nödvändigtvis avspeglar deluppgiftens svårhetsgrad. Uppgifterna kan besvaras på svenska eller engelska.

Behandla högst en uppgift per papper (det går bra att behandla deluppgifter på samma papper). Skriv bara på ena sidan och markera varje sida med dina initialer. Skriv läsligt.

▶ a)[0] Hur många uppgifter får man behandla per papper?

▶ b)[0] Får man skriva på baksidan?

Det går bra att referera till kursboken av Rosen eller Husfeldts anteckningar. Referenser måste ha formen [R: *något*] eller [H: *något*], där *något* är ett avsnittsnummer, en kodbit, en räknad uppgift, etc. Skriv till exempel "genom att använda Lemma [R:7.4]".

Exercise 1 (Logic)

This exercise studies a new quantifier, $\exists_2$.

Let $\exists_2 x\, P(x)$ denote the quantification that there exist precisely two (no more, no less) x in the universe of discourse such that $P(x)$ holds.

Determine the truth value of the following statements, where the universe of discourse is the integers. In each case, give a short reason for your answer (not a formal proof).

- ▶ a)[1] $\exists_2 x\, (x^2 = 1)$
- ▶ b)[1] $\exists_2 x\, (x = -x)$

Determine the truth value of the following statements. In each case, give a short reason for your answer (not a formal proof).

- ▶ c)[1] $\exists x P(x) \rightarrow \exists_2 x P(x)$
- ▶ d)[1] $\big(\exists x P(x) \wedge \neg \forall x P(x)\big) \rightarrow \exists_2 x P(x)$

The following two questions are concerned with replacing the new quantifier $\exists_2$ by the usual ones.

- ▶ e)[3] In this question (and only here) assume that the universe of discourse is $\{a, b, c\}$. Express the statement $\exists_2 x P(x)$ using only the logical operators negation, conjunction, and disjunction.
- ▶ f)[3] Express the statement $\exists_2 x P(x)$ using only $\forall$ and $\exists$ and the logical operators negation, conjunction, and disjunction.

Exercise 2 (Combinatorics)

In this exercise k and n are integers with $n > 0$ and $0 \leq k \leq n$.

The *Stirling number of the second kind* is written as $\left\{ {n \atop k} \right\}$. It denotes the number of different ways to partition the set $\{1, 2, \ldots, n\}$ into k nonempty subsets.

For example, $\{1, 2, 3, 4\}$ can be partitioned into 2 nonempty subsets in 7 ways:

$$\begin{array}{llll} \{1,2,3\} \cup \{4\}, & \{1,2,4\} \cup \{3\}, & \{1,3,4\} \cup \{2\}, & \{2,3,4\} \cup \{1\}, \\ \{1,2\} \cup \{3,4\}, & \{1,3\} \cup \{2,4\}, & \{1,4\} \cup \{2,3\}, & \end{array}$$

which shows that $\left\{ {4 \atop 2} \right\} = 7$.

Things to note:

1. the partition $\{1, 2, 3, 4\} \cup \emptyset$ is not on the list (we want *nonempty* subsets).
2. partitions like $\{1, 3, 2\} \cup \{4\}$ and $\{4\} \cup \{1, 2, 3\}$ are not on the list (they are considered the same as $\{1, 2, 3\} \cup \{4\}$).

Also, to remove any doubts, we define $\left\{ {n \atop 0} \right\} = 0$ and $\left\{ {n \atop n} \right\} = 1$.

► a)[1] Find $\left\{ {4 \atop 3} \right\}$ by listing all ways to partition $\{1, 2, 3, 4\}$ into 3 nonempty subsets.

► b)[1] Find $\left\{ {5 \atop 2} \right\}$.

► c)[3] Find a formula for $\left\{ {n \atop 2} \right\}$ for $n \geq 2$.

► d)[2] Give a combinatorial argument for the identity

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\},$$

when $k > 0$, $n > 1$.

► e)[3] (Difficult!) Prove by induction on n that if $x > n$ is an integer then

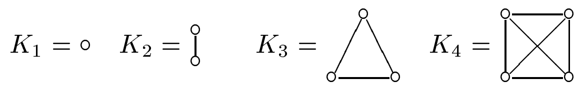
$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{x!}{(x-k)!} = x^n.$$

Hint: There are many ways to solve solve this problem, but you may find the following manipulation (which is not hard to prove) useful:

$$x \frac{x!}{(x-k)!} = \frac{x!}{(x-k-1)!} + k \frac{x!}{(x-k)!}$$

Exercise 3 (Recursion and Recurrences)

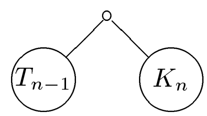
Recall that the n-clique K_n is the undirected graph with n nodes and all connecting edges, for example



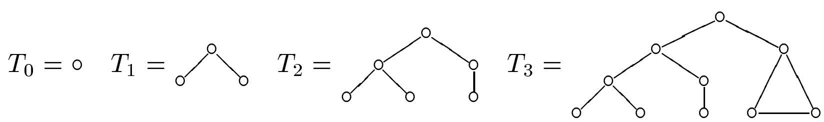
Note that the number of edges in K_n is $\binom{n}{2} = \frac{1}{2}n(n-1)$.

This exercise studies a now family of graphs, which we will call *cliquetree*. The nth *cliquetree* T_n is defined recursively as follows.

1. T_0 consists of a single vertex r, called the root, and nothing else.
2. T_n consists of a root r with two neighbours: r is connected to the root of T_{n-1} and to some vertex of K_n:



Here are the first few cliquetrees:



► a)[1] Draw T_4 and T_5

► b)[2] Let v_n denote the number of vertices of T_n. Find a recurrence relation for v_n.

► c)[3] Solve the recurrence relation that you found in the previous question. (If you don't have an answer for the previous question, you can solve $a_0 = 1$, $a_n = 2a_{n-1} + n + 3$ instead.)

► d)[1] Let e_n denote the number of edges of T_n. Find a recurrence relation for e_n.

► e)[3] Prove that $e_n = \frac{1}{6}n^3 + \frac{11}{6}n$.