Prov på delkurs Diskret Matematik

17 november 2001 kl 8–13 MA:9

Tillåtna hjälpmedel: Litteratur, egna anteckningar

Tentamen består av 3 uppgifter, varje uppgift kan ge 10 poäng. Deluppgifternas poäng anges inom [hakparenteser]. Observera att poänggivningen inte nödvändigtvis avspeglar deluppgiftens svårhetsgrad. Uppgifterna kan besvaras på svenska eller engelska.

Behandla högst en uppgift per papper (det går bra att behandla deluppgifter på samma papper). Skriv bara på ena sidan och markera varje sida med dina initialer. Skriv läsligt.

▶ a)[0] Hur många uppgifter får man behandla per papper?

▶ b)[0] Får man skriva på baksidan?

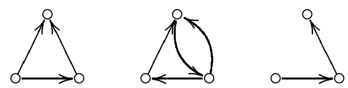
Det går bra att referera till kursboken av Rosen eller Husfeldts anteckningar. Referenser måste ha formen [R: *något*] eller [H: *något*], där *något* är ett avsnittsnummer, en kodbit, en räknad uppgift, etc. Skriv till exempel "genom att använda Lemma [R:7.4]".

Exercise 1 (Graphs, Probability)

A *tournament* is a directed graph $T = (V, E)$ where

- for every vertex $v \in V$ we have $(v, v) \notin E$ (no loops)
- for every pair of distinct vertices $u, v \in V$ ($u \neq v$) we have either $(u, v) \in E$ or $(v, u) \in E$, but not both.

Of the next following three graphs, only the first is a tournament:



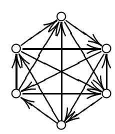
The name tournament is natural, since one can think of the set V as a set of players in which each pair participates in a single match, where $(u, v) \in E$ means that u beats v.

▶ a)[1] Draw a tournament on 4 players.

▶ b)[1] Find the number of different tournaments on 4 players, and on n players.

A tournament is *pair-dominating* if for every pair of players u, v there is another player who beats them both.

▶ c)[1] Is the following tournament on 6 players pair-dominating?



▶ d)[1] Is there a pair-dominating tournament on 5 players? If yes, draw it. If no, write 'no.'

A *random tournament* is constructed as follows. Flip a coin for every pair $\{u, v\}$ of nodes and insert the edge (u, v) if the outcome is 'tail', otherwise insert the edge (v, u).

▶ e)[1] Find the probability that player 1 wins all his matches.

▶ f)[1] Find the probability that no player beats both player 1 and player 2.

▶ g)[4] Assume that n is large enough so that

$$\binom{n}{2}\left(\frac{3}{4}\right)^{n-2} < 1.$$

Show that a random tournament on n players is pair-dominating with nonzero probability (this shows that such a tournament exists).

Exercise 2 (Combinatorics)

This exercise considers permutations $p_1 p_2 \cdots p_n$ of the numbers $\{1, 2, \ldots, n\}$, i.e., sequences where each number appears exactly once. A *step-up* is an index k where the sequence 'steps up': $p_k < p_{k+1}$. Here are the step-ups for an example permutation

$$9\,8\,\underline{5}\,\underline{6}\,7\,3\,2\,\underline{1}\,5\,4,$$

where we have marked the step-ups by underlining them. The following 11 permutations of $\{1, 2, 3, 4\}$ have exactly 2 step-ups:

$$\begin{array}{lllll}
\underline{1}\,3\,\underline{2}\,4 & \underline{1}\,4\,\underline{2}\,3 & \underline{2}\,3\,\underline{1}\,4 & \underline{2}\,4\,\underline{1}\,3 & \underline{3}\,4\,\underline{1}\,2 \\
\underline{1}\,\underline{2}\,4\,3 & \underline{1}\,\underline{3}\,4\,2 & \underline{2}\,\underline{3}\,4\,1 & & \\
2\,\underline{1}\,\underline{3}\,4 & 3\,\underline{1}\,\underline{2}\,4 & 4\,\underline{1}\,\underline{2}\,3 & &
\end{array}$$

Based on this, we define the step-number $\genfrac{\langle}{\rangle}{0pt}{}{n}{k}$ to be the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k step-ups. We just saw $\genfrac{\langle}{\rangle}{0pt}{}{4}{2} = 11$.

Just to avoid confusion, let us agree that

$$\genfrac{\langle}{\rangle}{0pt}{}{0}{k} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k = 1. \end{cases}$$

▶ a)[1] Show $\genfrac{\langle}{\rangle}{0pt}{}{3}{1} = 4$ by listing all 4 permutations of $\{1, 2, 3\}$ with 1 step-up.

▶ b)[1] Find $\genfrac{\langle}{\rangle}{0pt}{}{4}{1}$.

▶ c)[1] Is it true that

$$\genfrac{\langle}{\rangle}{0pt}{}{n}{k} \stackrel{?}{=} \genfrac{\langle}{\rangle}{0pt}{}{n}{n-k}.$$

If yes, give a combinatorial argument. If no, give a counterexample.

▶ d)[2] Is it true thata

$$\genfrac{\langle}{\rangle}{0pt}{}{n}{k} \stackrel{?}{=} \genfrac{\langle}{\rangle}{0pt}{}{n}{n-1-k}.$$

If yes, give a combinatorial argument. If no, give a counterexample.

▶ e)[2] (Harder) Give a combinatorial argument for the recurrence

$$\genfrac{\langle}{\rangle}{0pt}{}{n}{k} = (k+1)\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k} + (n-k)\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k-1}.$$

▶ f)[3] (Easier than it looks) Prove the following impressive identity about step-up numbers and binomial coefficients

$$x^n = \sum_{k=0}^{n} \genfrac{\langle}{\rangle}{0pt}{}{n}{k} \binom{x+k}{n} \qquad (\text{integers}^\dagger\ x \geq n \geq 0).$$

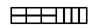
by induction in n. You may use the recurrence from the previous exercise. *Hint.* First prove (not by induction!) that

$$x\binom{x+k}{n} = (k+1)\binom{x+k}{n+1} + (n-k)\binom{x+k+1}{n+1}.$$

†Incidentally, the formula is true for all real x (the same proof works) but since we only defined the binomial coefficients for integers we restrict our attention to those.

Exercise 3 (Recurrences)

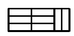
A *domino tile* is a 2×1 piece that can have two different orientations: or . A 2×10 area can be *covered* with dominoes in many ways, for example

 or 

For $n \geq 1$, let a_n be the number of different ways to cover a $2 \times n$ area with dominoes. For example, $a_2 = 2$ because there are only two ways to cover a 2×2 area:  and .

- ▶ a)[1] Find a_1 and a_3.
- ▶ b)[3] Find a recurrence relation for a_n.
- ▶ c)[3] Solve the recurrence relation from the previous question. If you didn't solve the previous question, solve the following recurrence: $a_1 = 3$, $a_2 = 5$, $a_n = a_{n-1} + a_{n-2}$.

An m-domino has dimensions $m \times 1$ (so a domino is a 2-domino). Here is a covering of 3×8 with 3-dominioes:



- ▶ d)[3] Find a recurrence for $b_{n,m}$, the number of ways to cover an $m \times n$ area with m-dominoes. Solve the recurrence for $m = 3$.