Hardness Results for Dynamic Problems by Extensions of Fredman and Saks' Chronogram method*

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Abstract We introduce new models for dynamic computation based on the cell probe model of Fredman and Yao. We give these models access to nondeterministic queries or the right answer ± 1 as an oracle. We prove that for the dynamic partial sum problem, these new powers do not help, the problem retains its lower bound of $\Omega(\log n / \log \log n)$.

From these results we easily derive a large number of lower bounds of order $\Omega(\log n / \log \log n)$ for conventional dynamic models like the random access machine. We prove lower bounds for dynamic algorithms for reachability in directed graphs, planarity testing, planar point location, incremental parsing, fundamental data structure problems like maintaining the majority of the prefixes of a string of bits and range queries. We characterise the complexity of maintaining the value of any symmetric function on the prefixes of a bit string.

1 Introduction

Update versus query time. For dynamic problems, two trivial solutions are immediate: Either the algorithm spends time after each update reorganising the data structure to anticipate every future query, or the algorithm spends time after each query to read the entire history of updates. However, a crucial property of many hard problems is that these two cannot be optimised simultaneously. This tradeoff between *update time* and *query time* was studied using the *chronogram method* by Fredman and Saks [13], a result that has proved extremely useful for lower bounds for dynamic algorithm and data structures.

The method of [13] is an information-theoretic argument formalising the idea that not all relevant information about the updates can be passed on to a typical query. The present paper takes a closer look at this information, asking what kind of information is responsible for the hardness of the problem. Our approach is to provide the query algorithm with well-defined aspects of the information for free, e.g., we consider nondeterministic query algorithms.

Example: Range queries. We can illustrate our approach using range query problems. The object is to maintain a set $S \subseteq \{1, \ldots, n\}^2$ of points in the plane, the updates insert and remove points from S. An *existential* range query asks whether a given rectangle R contains a point from S. This problem requires time $\Omega(\log \log n / \log \log \log n)$ [4, 20].

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With nondeterministic queries, this problem becomes trivial: guess a point and verify that it is in $S \cap R$. In other words, the sole reason for the hardness of this problem lies in maintaining precisely the kind of information that nondeterminism provides for free. However, this is not true for all problems; our main result implies that reporting the *parity* of $|R \cap S|$ remains just as hard as without nondeterminism, so the hardness of this problem hinges on information of a fundamentally different kind.

Main contribution. We state our two main results in terms of the *signed partial sum problem*. The problem is to maintain a string $x \in \{-1, 0, +1\}^n$ under updates that change the letters of x and queries of the form

query(i): return $x_1 + \cdots + x_i \mod 2$.

We prove two theorems about this problem. Theorem 1 shows that even in models with nondeterministic queries, the partial sum problem requires time $\Omega(\log n/\log\log)$ per operation with logarithmic cell size. It is known that this is also the deterministic complexity of the problem [7, 13], so nondeterminism does not help.

Our second main result studies the same problem in a *promise* setting, where the query algorithm receives almost the correct answer for free. The updates are as before, and the query is

parity(i, s): return $x_1 + \cdots + x_i \mod 2$ provided that $|s - \sum_{j=1}^{i} x_j| \leq 1$ (otherwise the behaviour of the query algorithm is undefined).

Theorem 2 shows that this problem still requires $\Omega(\log n/\log\log n)$ per operation.

We reason within the cell probe model of Fredman [10] and Yao [28], with some extensions to cope with our stronger modes of computation. This can be viewed as a nonuniform version of the random access computer with arbitrary register instructions. Especially, our lower bounds are valid on random access machines with unit-cost instructions on logarithmic cell size. The success of this model is partly due to the validity of these bound in light of schemes like hashing, indirect addressing, bucketing, pointer manipulation, or recent algorithms that exploit the parallelism inherent in unit-cost instructions. For these reasons the cell probe model has arguably become the model of choice for lower bounds for dynamic computation.

Theorems 1 and 2 are proved by extending the chronogram method, which was introduced by Fredman and Saks [13] and got its name in [5].

Lower bounds for dynamic algorithms. Our results suggest a new general technique for proving lower bounds for dynamic algorithm and data structure problems. Because Thms. 1 and 2 hold in very strong models of computation, we can exploit these strengths in our reductions—this yields simple proofs. We support our claims about the versatility of this technique by exhibiting a number of new lower bounds for well-studied problems, including planar point location, reachability in upward planar digraphs and in grid graphs, incremental parsing of balanced parentheses, and partial sum problems.

Limitations of the chronogram method. A large number of hardness results for dynamic problems employ the chronogram method, usually by constructing a reduction from a partial sum problem. Our results imply in some precise sense that this method is unable to distinguish deterministic from nondeterministic computation. In particular, this method cannot prove lower bounds for a problem that are better than the best nondeterministic algorithm. This is an important guide in the search for lower bounds for a large class of problems, including for example existential range searching and convex hull.

Outline of paper. Section 2 introduces dynamic algorithms with nondeterministic queries and contains the statement of Thm. 1; the proof of this result, which is the main technical contribution of this paper, is sketched in Sect. 3. Our lower bounds for dynamic algorithms and partial sum problems are presented in Sect. 4. Finally, Sect. 5 introduces the notion of refinement and presents Thm. 2. Many proofs are omitted due to space limitations, they can be found in the full version [15].

2 Nondeterminism in Dynamic Algorithms

2.1 Nondeterministic query algorithms. We now introduce our notion of nondeterministic query algorithms for dynamic decision problems. We allow query algorithms to nondeterministically load a value into a memory cell. The semantics is as usual: The value returned by a nondeterministic query is 1 unless all nondeterministic choices return 0. For example, the following program solves the existential range query problem from the introduction, storing all points from S in a two-dimensional array M:

update(i, j):	*query*(R):
$M[i, j] := \neg M[i, j]$	**guess** $(i, j) \in R$
	return $M[i, j]$

We should mention that we have not defined the *side-effects* of a nondeterministic query algorithm, i.e., the effect of its assignments to memory. This can be done in a number of ways; for example we might say that if there are computations (i.e., sequences of nondeterministic choices) that result in '1', the algorithm will execute one of these computations; otherwise it will execute a computation leading to '0'. We mention that our lower bound is immune to precisely how these effects are defined, since the hard operation sequence constructed in the proof needs only a single query, which happens at the very end.

Nondeterministic queries are a powerful tool for a number of well-studied problems. A good example from Computational Geometry is *dynamic convex hull*, the problem of maintaining the convex hull of a set of points S, where points are inserted and removed. The query operation asks whether the query point q lies inside or outside the convex hull of S. Again, we can solve this problem with a trivial update algorithm that simply stores S in a large table (in the cell probe model we do not worry about memory space, otherwise we can use standard dictionaries). The nondeterministic query guesses three points from S and verifies that the query point lies in the triangle spanned by these points—a well known result in plane geometry asserts that this is necessary and sufficient.

Note that the complement of this problem (answer 'yes' iff q lies outside the convex hull) does not seem to allow such an algorithm. In contrast, the complement of the existential range query problem in one dimension does, since we can maintain a doubly linked list of the inserted points, and the query can guess both the immediate predecessor and immediate successor of a query interval and verify that they are neighbours is S.

In general, a problem is amenable to nondeterminism, if the outcome of each query depends on only a bounded number of updates. Contrast this with the problems identified in [13], where each update affects only a bounded number of queries, e.g., dictionary problems.

2.2 Signed partial sum. The *signed partial sum* problem is to maintain a string $x \in \{-1, 0, +1\}^n$, initially 0^n, under updates that change the letters of x and queries about the parity of the prefix sums of x

update(i, a): change x_i to $a \in \{-1, 0, +1\}$, *query*(i): return $x_1 + \cdots + x_i \mod 2$.

The data structure of Dietz [7] solves this problem, deterministically, in time $O(\log n/\log\log n)$ per operation with logarithmic cell size. The next theorem states that nondeterministic queries can do no better. We state theorem as a trade-off between update and query time.

Theorem 1 *Every nondeterministic algorithm for the signed partial sum problem with cell size b, update time t_u, and query time t_q must satisfy*

$$t_q = \Omega\Big(\frac{\log n}{\log(bt_u \log n)}\Big). \tag{1}$$

The lower bound holds even if the algorithm requires

$$0 \le x_1 + \cdots + x_i \le \Big\lceil \frac{\log n}{\log(bt_u \log n)} \Big\rceil \tag{2}$$

for all i after each update.

The balancing condition (2) continues previous work [16] on extending the chronogram method, which is implicit in the constructions in the present paper. In Sect. 4.2 we state a further generalisation of Thm. 1, relating the terms in (1) and (2).

3 Proof of Theorem 1

We consider a specific sequence of operations that consists of a number of updates followed by a single query. The update sequence is chosen at random from a set U defined in Sect. 3.5.

3.1 Model of computation. The computational model is an extension of the cell-probe model [10, 28]; since there is only a single query in the hard sequence of operations constructed in our proof, which happens at the very end of the sequence, we can model query algorithms by nondeterministic decision trees.

More precisely, a *cell probe* algorithm consists of a family of trees, one for each operation, and a memory $M \in \{0, \ldots, 2^b - 1\}^*$. We refer to the elements of M as *cells*, each of which can store a b-bit number. To each update we associate a decision–assignment tree as in [13]. There are two types of nodes: *Read* nodes are 2^b-ary and labelled by a memory address, computation proceeds to the child identified at that address; *write* nodes are unary and labelled by a memory address and a b-bit value, with the obvious semantics.

To each query we associate a nondeterministic decision tree of arity 2^b whose internal nodes are labelled by a memory address or by '$\exists$'. The leaves are labelled 0 or 1 to represent the possible answers to the query. We define the value $qM \in \{0, 1\}$ computed by a query tree q on memory M to be 1 if there exists a path from the root to a leaf with label 1. A witness of such an accepting computation is the description of the choices for the $\exists$ nodes. We let q_i denote the query tree corresponding to *query*(i). The query time t_q is the height of the largest query tree and the update time t_u is the height of the largest update tree; we account only for memory reads and writes and for nondeterministic choices, all other computation is for free.

3.2 Updates and epochs. Each update sequence in U is described by a binary string $u \in \{0, 1\}^*$. Each bit represents an update *update*(j, a). The parameters for these updates will be specified in Sect. 3.5. The update sequences $u \in U$ are split into d substrings each corresponding to an *epoch*. It turns out to be convenient that time flows backwards, so epoch 1 corresponds to the end of u. In general the update string is an element in $U = U_d U_{d-1} \cdots U_1$ where $U_t = \{0,1\}^{e(t)}$, and where $e(t)$ is the length of epoch t is such that $e(t)+\cdots+e(1) = \lfloor n^{t/d}/d \rfloor$. The length of the entire update sequence is $\lfloor n/d \rfloor$. The size of d and hence the growth rate of $e(t)$ is $d = \lceil \log n / \log(b t_{\mathrm{u}} \log n) \rceil$. The goal is to establish that $t_{\mathrm{q}} \in \Omega(d)$.

3.3 Time stamps and nondeterminism. To each cell we associate a time stamp when it is written. A cell receives time stamp t if some update during epoch t writes to it, and none of the subsequent updates during epochs $t-1$ to 1 write to it.

For an update sequence $u \in U$ let M^u denote the memory resulting from these updates (recall that updates are restricted to perform deterministically), starting with some arbitrary initial contents corresponding to the initial instance 0^n.

For index i and update string u let $T(i,u)$ denote the set of time stamps that are found on every accepting computation path of q_i on M^u. If there are no accepting computations, the set is empty. More formally, let w denote a witness for a computation path of q_i on M^u, and let $A(i,u)$ denote the set of witnesses that lead to accepting computations of q_i on M^u. Let for a moment $T(i,u,w)$ denote the set of time stamps encountered by the computation of q_i on M^u that is witnessed by w. Then $T(i,u) = \bigcap\{\, T(i,u,w) \mid w \in A(i,u) \,\}$ if $A(i,u) \neq \emptyset$, and $T(i,u) = \emptyset$ otherwise.

The simple lemma below is the tool to identify a read of a cell with time stamp t by nondeterministic queries.

Lemma 1 *If M^u and M^v differ only on cells with time stamp t then $q_i M^u \neq q_i M^v$ implies $t \in T(i,u) \cup T(i,v)$.*

3.4 Lower bound on query time. The update sequences are chosen such that even if two sequences differ only in a single epoch, they still result in very different instances. To each update sequence $u \in U$ we associate the query vector $q^u = (q_1 M^u, q_2 M^u, \ldots, q_n M^u) \in \{0,1\}^n$. Update sequences that differ only in epoch t are called *t-different.*

Lemma 2 *No Hamming ball of diameter $\frac{1}{8}n$ can contain more than $|U_t|^{9/10}$ query vectors from t-different update sequences, for large n.*

The difficult part is constructing a set of update sequences for which the statement is true, which we present in Sect. 3.5. The proof itself is as in [13].

Write $U_{>t}$ for $U_d \cdots U_{t+1}$, the set of updates sequences prior to epoch t, and $U_{<t}$ for $U_{t-1} \cdots U_1$, the set of update sequences in epoch t to epoch 1. Assume for the rest of this section that $t_{\mathrm{q}} = \mathrm{O}(\log n)$, else there is nothing to prove. The worst-case query time t_{q} is at least the average of $|T(i,u)|$ over choices of $i \in \{1,\ldots,n\}$ and $u \in U$, so

$$|U| n t_{\mathrm{q}} \geq \sum_{u \in U} \sum_{i=1}^{n} |T(i,u)| = \sum_{t=1}^{d} \sum_{u \in U_{>t}} \sum_{w \in U_{<t}} \sum_{v \in U_t} \sum_{i=1}^{n} \big(t \in T(i, uvw)\big).$$

The next lemma tells us how many $v \in U_t$ fail to make the last sum exceed $\frac{1}{16}n$.

Lemma 3 *Fix any epoch $1 \leq t \leq d$ and past and future updates $x \in U_{<t}$, $y \in U_{>t}$. For large n, at least half of the update sequences $u \in x U_t y$ satisfy $\big|\{\, 1 \leq i \leq n \mid t \in T(i,u) \,\}\big| \geq \frac{1}{16}n$, if $t_{\mathrm{q}} = \mathrm{O}(\log n)$.*

By this lemma we obtain for large n:

$$|U| n t_{\mathrm{q}} \geq \sum_{t=1}^{d} |U_{>t}| \cdot |U_{<t}| \cdot \tfrac{1}{16} n \cdot \tfrac{1}{2} |U_t| = \tfrac{1}{32} n d |U|,$$

and hence $t_{\mathrm{q}} \geq \frac{1}{32} d$ as desired.

3.5 Update scheme. The technical part that remains is to exhibit a set of update sequences U satisfying Lem. 2. There are a number of ways to do this; the following construction is one which simultaneously anticipates our needs in Sect. 5 and satisfies the balancing condition (2).

To alleviate notation we assume that n/d is an integer. Consider the updates in epoch t and index them as $u_1 \cdots u_{e(t)} \in U_t$. If $u_i = 0$ then nothing happens in the ith update. Else it performs $update(j, a)$, where the update position j is given below. The new value is $a = (-1)^r$, where $r = 1 + u_1 + \cdots + u_i \bmod 2$, so the nonzero updates in u alternate between -1 and $+1$, starting with $+1$. The position of the affected letter is defined as follows. Write x as a table of dimension $d \times n/d$ like this:

$$\begin{bmatrix} x_1 & x_{d+1} & & x_{n-d+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_d & x_{2d} & & x_n \end{bmatrix} .$$

All updates in epoch t will affect only the letters in row t. The updates of an epoch are spread out evenly from left to right across that row, so the distance between two of them is $\lfloor (n/d)/e(t) \rfloor$. In summary, the ith update in epoch t affects the letter in row t and the column given by $(i-1) \cdot \lfloor (n/d)/e(t) \rfloor + 1$.

This update scheme satisfies the statement in Lem. 2, we omit the proof. Also, the prefix sums of instances resulting from our scheme are small: Let x denote an instance resulting from our scheme from the initial instance 0^n. Let x^t denote the string resulting from only the updates in epoch t and write x as $x^1 + \cdots + x^d$; this works because no two epochs write in the same positions. Then

$$\sum_{j=1}^{i} x_j = \sum_{j=1}^{i} \sum_{t=1}^{d} x_j^t = \sum_{t=1}^{d} \sum_{j=1}^{i} x_j^t \in \{0, \ldots, d\} ,$$

because the prefix sum of every x^t is 0 or 1 by construction. It can be checked that the balancing bound (2) holds at all times.

Another important feature of this update scheme, which is used to prove Thm. 2, is that if x and y result from t-different updates then $x^r = y^r$ for $r \neq t$ and hence $\left| \sum_{j=1}^{i} x_j - \sum_{j=1}^{i} y_j \right| \leq 1$ for all i.

4 Lower Bounds for Dynamic Algorithms and Partial Sum Problems

Theorem 1 suggests a new approach for proving lower bounds by employing nondeterminism in the reduction from signed partial sum. We demonstrate this with a number of examples in this section. The results are presented for cell size $b = \log n$ for concreteness. Some of the reductions extend previous work of the authors with Søren Skyum [16].

4.1 Nested brackets. Consider the problem of maintaining a nested structure, i.e., a string x with round and square brackets under the following operations:

change(i, a): change x_i to a, where a is a round or square opening or closing bracket, or whitespace.

balance: return 'yes' if and only if the brackets in x are properly nested.

This problem was studied in [9], where an algorithm with polylogarithmic update time is presented.

Proposition 1 *Maintaining a string of nested brackets requires time* $\Omega(\log n / \log\log n)$ *per operation.*

Proof. Consider a deterministic algorithm for this problem and let $x \in \{0, -1, +1\}^n$ be an instance to signed partial sum. Let b_i be an encoding of x_i given by:

$$+1 \mapsto))\sqcup, \quad 0 \mapsto)\sqcup\sqcup, \quad -1 \mapsto \sqcup\sqcup\sqcup,$$

where '$\sqcup$' stands for space. Let c be the string '$\sqcup($'. We maintain a balanced string of brackets uvw, where $u = c^{2n}$, $v = b_1 b_2 \ldots b_n$ and $w =)^{n-s} \sqcup^s$, where $s = x_1 + \cdots + x_n$. It is easy to see that uvw balances and can be maintained by a constant number of updates per update in x. For any prefix size i this construction enables efficient verification of a nondeterministic guess g of the prefix sum $x_1 + \cdots + x_i$: Place a closing square bracket on the last $\sqcup$ of b_i and an opening square bracket on the $\sqcup$ of the first c of suffix c^{i+g} of u. This modification keeps uvw balanced iff g is the right guess of prefix sum $x_1 + \cdots + x_i$. Conclusion by Thm. 1. ∎

4.2 Dynamic graph algorithms. Our techniques improve the lower bounds of a number of well-studied graph problems considered in [16].

Tamassia and Preparata [26] present an algorithm for the class of *upward planar source–sink graphs* that runs in time $O(\log n)$ per operation. These digraphs have have a planar embedding where all edges point upward (meaning that their projection on some fixed direction is positive) and where exactly one node has indegree 0 (the source) and exactly one node has outdegree 0 (the sink). The updates are:

insert(u, v): insert an edge from u to v
delete(u, v): delete the edge from u to v if it exists
reachable(u, v): return 'yes' iff there is a path from u to v.

The updates have to preserve the topology of the graph, including the embedding.

Proposition 2 *Dynamic reachability in upward planar source–sink graphs requires time* $\Omega(\log n / \log \log n)$ *per operation.*

Planarity testing is to maintain a planar graph where the query asks whether a new edge violates the planarity of the graph. Italiano *et al.* [18] present an efficient algorithm for a version of this problem, and a strong lower bound is exhibited by Henzinger and Fredman [12]. Our lower bound holds also for *upward* planarity testing, where the topology is further restricted to upward planar graphs. The updates insert and delete edges as above, and the query is

planar(u, v): return 'yes' if and only if the graph remains upward planar after insertion of edge (u, v).

This problem was studied by Tamassia [25], who found an $O(\log n)$ upper bound.

Proposition 3 *Upward planarity testing requires time* $\Omega(\log n / \log \log n)$ *per operation.*

A classical problem in Computational Geometry is *planar point location*: given a subdivision of the plane, i.e., a partition into polygonal regions induced by the straight-line embedding of a planar graph, determine the region of query point $q \in \mathbf{R}^2$. An important restriction of the problem considers only *monotone* subdivisions, where the subdivision consists of polygons that are monotone (so no horizontal line crosses any polygon more than twice). In the dynamic version of this problem updates manipulate the geometry of the subdivision. Preparata and Tamassia [24] give an algorithm that runs in time $O(\log^2 n)$ per operation, this was improved to query time $O(\log n)$ by Baumgarten, Jung, and Mehlhorn [3]. The lower bound for this problem in [16] applies only to algorithms returning the name of the region containing the queried point. The techniques of the present paper extend this bound to work for simpler decision queries like

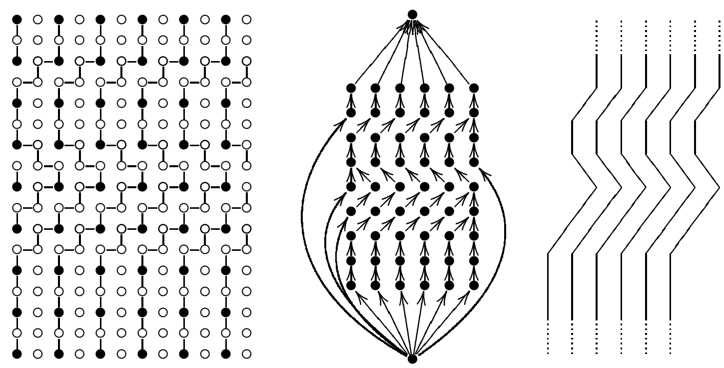


Fig. 1. Planar graphs corresponding to $x = (0,\ 0,\ +1,\ +1,\ -1,\ 0,\ +1,\ 0)$. Left: grid graph. Even grid points are marked $\bullet$, odd grid points are marked $\circ$. Middle: upward planar source–sink graph. Right: monotone planar subdivision.

query(x): return 'yes' if and only if x is in the same polygon as the origin.

Proposition 4 *Planar point location requires time* $\Omega(\log n/\log\log n)$ *per operation, even in monotone subdivisions.*

Traditionally, lower bounds in Computational Geometry are proved in an algebraic, comparison-based model (see [23] for a textbook account) that is broken by standard RAM operations like indirect addressing, bucketing, hashing, etc. Cell probe lower bounds for that field are lacking.

To explain our reduction we turn to the conceptually very simple class of *grid graphs*. The vertices of a grid graph of width w and height h are integer points (i, j) in the plane for $1 \le i \le w$ and $1 \le j \le h$. All edges have length 1 and are parallel to the axes. The dynamic reachability problem for these graphs is the following:

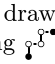
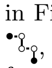
flip(x, y): add an edge between $x \in [w] \times [h]$ and $y \in [w] \times [h]$ or remove it if it exists,

reachable(x, y): return 'yes' if and only if there is a path from x to y.

There are several well-known constructions that prove a lower bound for this problem [8, 12, 14, 21], but our proof translates to the other problems in Props. 2 to 4. The details in these constructions are omitted, Fig. 1 illustrates the structures arising in the reductions.

Proposition 5 *Dynamic reachability in grid graphs requires time* $\Omega(\log n/\log\log n)$ *per operation.*

Proof. From an instance $x \in \{0, \pm 1\}^n$ to signed partial sum we build a grid graph on the points $\{0, \dots, 2w\} \times \{0, \dots, 2n\}$, where $w = \lceil \log n/\log\log n \rceil$. We will exploit the balancing constraint (2) of Thm. 1 to keep the instance within this width.

For every i and j, consider any point with even coordinates $(2i, 2j - 2)$, drawn as $\bullet$ in Fig. 1, and connect it to one of the three even grid points above it using , , or , depending on whether $x_j = +1$, 0, or -1, respectively. The idea is that the path from $(0, 0)$ mimics the prefix sums of x in that it passes through $(2s, 2j)$ if and only if

$x_1 + \cdots + x_j$ equals s. Hence a guess of the sum can be verified by a single reachability query in the graph.

It remains to note that the graph can be maintained efficiently. Any changed letter in x incurs $\mathrm{O}(w)$ edges to be inserted or deleted. So if the update time of the graph algorithm is polylogarithmic then the graph can be maintained in polylogarithmic time. The bound follows from Thm. 1. ∎

The width of the hard graph above is logarithmic in the height, while the graphs constructed in [8, 12, 14, 21] are square. Hence narrow grid graphs are as hard as square ones. However, this is not true for *very* narrow graphs: It is known that the reachability problem for grid graphs of *constant* width can be solved in time $\mathrm{O}(\log\log n)$ by [2], an exponential improvement. This leaves open the question of what happens for graphs of sublogarithmic width. To answer this, we introduce a subtler statement of Thm. 1.

Theorem 1 (Parameterised) *Let* $d = \mathrm{O}\big(\log n/\log(bt_\mathrm{u}\log n)\big)$ *be an integer function. Every nondeterministic algorithm for signed partial sum with cell size* b*, update time* t_u*, and query time* t_q *must satisfy* $t_\mathrm{q} = \Omega(d)$*. The lower bound holds even if the algorithm requires* $0 \le x_1 + \cdots + x_i \le d$ *for all* i *after each update.*

This result implies a lower bound for grid graphs that smoothly connects the two extremes between linear and constant width. A similar parameterisation can be done for all our problems.

Proposition 6 *For every* $w = \mathrm{O}(\log n/\log\log n)$*, dynamic reachability in grid graphs of width* w *requires time* $\Omega(w)$ *per operation.*

4.3 Partial sum problems. The partial sum problem [11, 29] is to maintain a bit string $x \in \{0,1\}^n$ under the following operations

update(i): change x_i to $1 - x_i$,

sum(i): return $x_1 + \cdots + x_i$.

It was shown in [13] that the parity query

parity(i): return $x_1 + \cdots + x_i \mod 2$,

requires time $\Omega(\log n/\log\log n)$, so even the least significant bit is hard to maintain. We turn to two other natural variants, *prefix majority* and *prefix equality* whose query operations are

majority(i): return 1 iff $x_1 + \cdots + x_i \ge \lceil \frac{1}{2}n \rceil$,

equality(i): return 1 iff $x_1 + \cdots + x_i = \lceil \frac{1}{2}n \rceil$.

These problems arise in many data structures, e.g. when following paths towards heavy subtrees in balanced search trees. We can also dress up these problems as database queries like 'did as many male as female guests arrive before noon?' or 'are more French than English talks scheduled between Tuesday and Friday?' Similarly, these problems can be viewed as natural *range query* problems in Computational Geometry.

No nontrivial lower bounds for these two problems follow from [13]. The results from [4, 19, 20, 27] can be seen to imply $\Omega(\log\log n/\log\log\log n)$ lower bounds using an entirely different technique based on Ajtai's result [1]; and [16] reports $\Omega((\log n/\log\log n)^{1/2})$ for equality and $\Omega(\log n/(\log\log n)^2)$ for the majority.

The next result shows that these problems are just as hard as the parity query from [13]. The proof is again a simple application of Thm. 1.

Proposition 7 *The prefix equality and prefix majority problems require time* $\Omega(\log n/\log\log n)$ *per operation.*

There are other partial sum problems that are far easier. Consider the query

or(i): return 'yes' iff $x_1 + \cdots + x_i \geq 1$.

This problem, *prefix-or*, can be solved in time $O(\log\log n)$ per operation by a van Emde Boas tree. To study this kind of problem in a general, let the *threshold* ϑ be an integer function such that $\vartheta(i) \in \{0, \ldots, \lceil\frac{1}{2}i\rceil\}$. The query in the *prefix threshold problem for* ϑ is

threshold(i): return 'yes' iff $x_1 + \cdots + x_i \geq \vartheta(i)$.

Prefix majority is the special case $\vartheta(i) = \lceil\frac{1}{2}i\rceil$, prefix-or is $\vartheta(i) = 1$. Now for our lower bound. Our assumption on ϑ is that there are integers $p(1) < p(2) < \cdots < p(i) < \cdots$ such that $\vartheta(p(i)) = i$. We call such functions *nice* for lack of a better word. It is reasonable to assume that ϑ is monotonically increasing, the niceness assumption also prevents it from skipping points.

Proposition 8 *Let $t_\mathrm{u} = t_\mathrm{u}(n)$ and $t_\mathrm{q} = t_\mathrm{q}(n)$ denote the update and query time of any cell size b implementation of the prefix threshold problem for a nice threshold ϑ. Then $t_\mathrm{q} = \Omega\big(\log\vartheta/\log(t_\mathrm{u} b \log\vartheta)\big)$.*

The proof is not difficult but tedious. The idea is to stretch an instance for a threshold problem, padding it with sufficiently many 0s or 1s to turn it into a majority problem.

To gauge the strength of this result we mention that the problem can be solved on the unit-cost RAM with logarithmic cell-size in time $O\big((\log\vartheta/\log\log n) + \log\log n\big)$ per update (if $\vartheta(1), \ldots, \vartheta(n)$ can be computed in the preprocessing stage of the algorithm). The left term in the expression stems from a search tree, the right term from a priority queue, which vanishes for cell size $b = \Omega(\log^2 n)$; details are omitted. Comparison with Prop. 8 shows that the lower bound is tight for logarithmic cell size and $\vartheta = \Omega(\log^{\log\log n} n)$. For smaller thresholds, the bounds leave a gap of size $O(\log\log n)$. We consider a more general problem in Sect. 5.1.

5 Refinement

We now take a somewhat subtler approach to our basic question than in Sect. 2. Instead of nondeterminism, we study the performance of query algorithms in a *promise* setting. We assume that the query algorithm for signed partial sum receives a value s that is promised to be *close to* (but not known to be equal to) the right sum and then decides between right and wrong values.

The *partial sum refinement* problem can be phrased as follows: Maintain a string $x \in \{0, \pm 1\}^n$, initially 0^n, under the following operations:

update(i, a): change x_i to $a \in \{-1, 0, +1\}$,

parity(i, s): return $x_1 + \cdots + x_i \mod 2$ provided that $|s - \sum_{j=1}^{i} x_j| \leq 1$ (otherwise the behaviour of the query algorithm is undefined).

The problem gets its name from the following alternative definition, where the query operation is replaced by

refine(i, s): return 1 if $s = \sum_{j=1}^{i} x_j$ and 0 if $s \neq \sum_{j=1}^{i} x_j$, provided that $|s - \sum_{j=1}^{i} x_j| \leq 1$. For other values of s, the answer is undefined.

The two problems reduce to each other.

Theorem 2 *Let d be an integer function such that $d = O\big(\log n/\log(t_\mathrm{u} b \log n)\big)$. Every algorithm for partial sum refinement with cell size b, update time t_u and query time t_q must satisfy $t_\mathrm{q} = \Omega(d)$. Moreover, this is true even for algorithms that require $0 \leq x_1 + \cdots + x_i \leq d$ for all i after each update.*

5.1 The dynamic prefix problem for symmetric functions. Thm. 2 acts as an important ingredient in characterising the dynamic complexity of all the *symmetric functions*, generalising the results for the threshold functions of last section. A Boolean function is *symmetric* if it depends only on the number of 1s in the input $x = (x_1, \ldots, x_n)$. The symmetric functions include some of the most well-studied functions in complexity theory, like parity, majority, and the threshold functions.

In general, we can describe every symmetric function f in n variables by its *spectrum*, a string in $\{0,1\}^{n+1}$ whose ith letter is the value of f on inputs where exactly i variables are 1. The *boundary* of a spectrum s is the smallest value ϑ such that $s_{\lfloor\vartheta\rfloor} = s_{\lfloor\vartheta\rfloor+1} = \cdots = s_{\lfloor n-\vartheta\rfloor}$. For instance the boundary of the parity or majority functions is $\frac{1}{2}n$, and for the threshold functions with threshold ϑ, the boundary is $\min(\vartheta, n-\vartheta)$.

Let $\langle f_n\rangle = (f_1, \ldots, f_n)$ be a sequence of symmetric Boolean function where the ith function f_i takes i variables. The *dynamic prefix problem for* $\langle f_n\rangle$ is to maintain a bit string $x \in \{0,1\}^n$ under the following operations:

update(i): change x_i to $\neg x_i$,

query(i): return $f_i(x_1, \ldots, x_i)$.

For example, taking f_i to be the parity function on i variables we have the prefix parity problem of [13], and taking f_i to be the threshold function for $\vartheta(i)$ we have the problem from Prop. 8.

Proposition 9 *Let ϑ be a nice function and let $\langle f_n\rangle$ be a sequence of symmetric functions where $f_i\colon \{0,1\}^i \to \{0,1\}$ has boundary $\vartheta(i)$. Let t_u and t_q denote the update and query time of any cell size b implementation of the dynamic prefix problem for $\langle f_n\rangle$. Then* $t_\mathrm{q} = \Omega\big(\log\vartheta / \log(t_\mathrm{u} b \log\vartheta)\big)$.

Intriguingly, the bound in the proposition is precisely the same bound as for the size–depth trade-off for Boolean circuits for these functions [17, 6, 22].

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